[13.40] Let V be an *n*-dimensional vector space,  the tensor product of V with itself, and tensor over V. Let  be the symmetric part and  be the antisymmetric part. Define  and . Then

1.  are vector spaces,
2. ,
3. ,
4. dim  and dim .

Proof. Penrose only asks us to show (4) and the reader can safely skip to that step now. However, I found (1-3) useful for enhancing my understanding of this topic since Penrose stated these without proof.

**Notation**.  and  where .

1. Let . Then[[1]](#footnote-1)

****

So V+ is closed under +. It is also closed under scalar multiplication:

.

Also,



and

 .

Since it is a subset of V, the remaining vector space axioms hold. ✔

1. Let . Since . Since . ✔
2. Let . Then , and . ✔
3. Let  be the basis for V where . Set

.

By definition,  is a basis for V, and it has *n*2 terms.

Define  and .

Observe that  and . So,  and thus  has  terms with *a* ≤ *b* and  has  terms with *a* < *b*.

Note: The reason for defining  and  with *a* ≤ *b* is that  and , so terms with *b* > *a* are not independent from the others.

Each set  and  is clearly consists of linearly independent vectors, so each constitutes a basis for a subspace of V. Moreover,  clearly is a basis for  and  is a basis for . ✔

An example with *n* = 2 may clarify part (4). Let  where  and . Then  and , and  is a basis for V. Next,  ,  , , and .

 is a basis for . Observe that  and . These are 2 elements of . Note that  so they do not contribute to . The other term in  is . The only term in  is . Thus dim  3 and dim  .

1. The proof of the line below is actually quite tricky [↑](#footnote-ref-1)