[13.40] Let V be an *n*-dimensional vector space,  the tensor product of V with itself, and let tensor. Let

 be the symmetric part

and

 be the antisymmetric part.

Define

 and .

Then

 and .

Solution.

Let  be the basis for V where . Set

.

By definition,  is a basis for V, and it has *n*2 terms. Observe that

 and .

So, we define

 and .

 has  terms with *a* ≤ *b* and  has  terms with *a* < *b*.

Note: The reason for defining  and  with *a* ≤ *b* and *a* < *b* is that



and

.

So, terms with *b* > *a* are not independent from the others.

Set . Claim is a basis for :

Consider two typical elements of ,



and

.

 are linearly independent because there is no scalar *a* such that . Moreover, is a linearly independent set because all other elements of have 0’s in the *a*-*b* and *b*-*a* positions. has dimension , and since has  independent elements, it is a basis for . ✔

Observe that dim span  and dim span .

We proceed to show that  and , which will complete the problem.

Claim: :

Let . When *a* ≤ *b*,  and 

 ✔

Claim: :

Let . When *a* < *b*,  and 

 ✔

Thus, .

Consider .

Denote . Then

.

Fix a < b:



Since , 

Therefore



 ✔

Similarly, for *a* < *b*,





 ✔ 

Note: Since , the set containing the zero matrix, then , the sum of disjoint subspaces.

Example with *n* = 2:

Let

 and .

Then  is a basis for V.

Let , ,

, and .

Then  is a basis for .

Observe that

 and .

These are 2 elements of .

Note that  so they do not contribute to .

The other term in  is .

The only term in  is  (which equals ).

Thus dim  3 and dim . ✔